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Indices of Banach Function Spaces and Spaces of Fundamental Type

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INTRODUCTION

Geometric poperties of abstract Banach function spaces, such as reflexivity and uniform convexity, were characterized by Halperin [14, 15, 20] in terms of topological conditions upon the norm of the space in question. For several concrete function spaces it is known (cf., e.g., [1, 14, 16, 20, 24]) that these properties are equivalent to certain growth conditions upon particular functions connected with the respective space.

The basic observation behind this paper is the fact that in case of Lebesgue spaces L_p much simpler conditions can be found, namely, in terms of the parameter p. For instance, it is well known that L_p is reflexive and uniformly convex if and only if $1 < p < \infty$. Our aim therefore is to find similar conditions also in the case of more general function spaces, namely, in terms of a double scale of reals, called indices, generalizing the number $1/p$. This aim is limited by the fact that there exist, e.g., nonreflexive and nonseparable rearrangement invariant Banach functions with indices strictly between 0 and 1 (playing the role of $0 < 1/p < 1$); see [18]. Nevertheless, we can still show that such conditions can be worked out for Orlicz and Lorentz spaces.

More specifically, Kamińska [16] proved that the Orlicz space $L_{M\Psi}$ is uniformly convex if and only if the Young function Ψ is uniformly convex and satisfies the Δ_2-condition. This typical result is extended by showing that the latter condition is equivalent to the lower fundamental index $\underline{\gamma}_{L_{M\Psi}}$ being strictly positive. In other words, the Orlicz space is uniformly convex if and only if Ψ is uniformly convex and $\underline{\gamma}_{L_{M\Psi}} > 0$. Similarly we proceed in case of reflexivity, separability, and absolute continuity of the norm. For Lorentz

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spaces $\Lambda(\phi, p)$ we extend a result of Halperin [14], for example, stating that $\Lambda(\phi, p)$ is uniformly convex if and only if

$$\sup_{t>0} \frac{\int_0^t \phi(u)\, du}{\int_0^{st} \phi(u)\, du} < 1$$

for some $s > 1$. Actually we obtain that the latter condition is equivalent to the index condition $\gamma_{\Lambda(\phi,p)} > 0$.

The crucial point is that these geometric properties are governed by the fundamental indices and not by the Boyd indices.

On the other hand, it is known that Boyd indices are useful in connection with mapping properties of the Hilbert transform (see [2]), with the Hardy property (see [5]), with the existence of a Schur–Hardy inequality, as well as with a Hilbert inequality concerning double integrals (see [6, 7]). Moreover, it is a "mixed condition" involving both fundamental indices and Boyd indices which is necessary and sufficient for the interpolation property of the space to hold (see [8]). Therefore, in the second part of this paper relations between fundamental indices and Boyd indices are studied, in particular the important problem as to when both kinds of indices do actually coincide.

These considerations lead us to the new definition of a space of fundamental type, a definition which should be of interest per se since it restricts the rather wide class of rearrangement invariant (r.i.) spaces to a more handy class which is still large enough to contain most of the spaces occurring in applications. In this connection it is also shown that spaces of fundamental type can be assigned to any r.i. space in a canonical way. Moreover, these considerations are the theoretical background for a forthcoming paper on interpolation [11].

1. Fundamental Indices and Growth Properties

In the sequel let (Ω, Σ, μ) denote a σ-finite, nonatomic measure space with $0 < \mu(\Omega) =: l \leqslant \infty$, $\mathscr{P}(\Omega)$ the set of all nonnegative, real-valued functions defined on Ω, ρ a rearrangement invariant (r.i.) function norm on $\mathscr{P}(\Omega)$, and $X := X_\rho(\Omega)$ the r.i. Banach function space generated by ρ (see, e.g. [22]). The associated r.i. Banach function space of X will be denoted by X'.

If $\Omega^* := (0, l)$, $l \leqslant \infty$, then there exists [21] a r.i. norm λ on the set $\mathscr{P}(\Omega^*)$ such that $\rho(f) = \lambda(f^*)$ for all $f \in \mathscr{P}(\Omega)$. Here f^* denotes the nonincreasing rearrangement of the function f. The r.i. function space $X_\lambda(\Omega^*)$ generated by this function norm λ will be called the *Luxemburg representation* of the space $X_\rho(\Omega)$. This representation makes it possible to define fundamental indices of the space $X_\rho(\Omega)$ as follows (for the case $\Omega = \Omega^*$, $\rho = \lambda$ see, e.g. [30]):

The *fundamental function* τ_X of the space $X := X_\rho(\Omega)$ is defined by $\tau_X(t) := \|\chi_{(0,\min\{t,l\})}\|_{X_\lambda(\Omega^*)}$, $t > 0$, where $\chi_{(0,t)}$ is the characteristic function of the interval $(0,t)$. Without loss of generality τ_X will be assumed to be concave.

The norm of the *dilation operator* E_s, $s > 0$, given by

$$(E_s f)(t) := f(st), \qquad st \in \Omega^*,$$
$$:= 0, \qquad \text{elsewhere},$$

namely, $h(s,X) := \|E_s\|_{[X_\lambda(\Omega^*)]}$, $s > 0$, is called the indicator function of the space $X = X_\rho(\Omega)$, see [3]. In [10] we showed that

$$\begin{aligned} \tau_X(st) &\leqslant h(1/s, X)\,\tau_X(t) && (t \in \Omega^*) \\ &\leqslant \tau_X(t) && (t \in (0,\infty)\backslash\Omega^*). \end{aligned} \tag{1.1}$$

On account of this inequality the function $M(\cdot, X)$, defined by

$$M(s,X) := \sup_{t,st\in\Omega^*} \frac{\tau_X(st)}{\tau_X(t)} \qquad (s > 0), \tag{1.2}$$

is finite-valued for each $s > 0$. So the following definition is meaningful.

DEFINITION 1.1. Let $X \equiv X_\rho(\Omega)$ be a r.i. Banach function space and $M(\cdot, X)$ as in (1.2). The *lower* and *upper fundamental indices* of the space X are, respectively, defined as

$$\underline{\gamma}_X := \sup_{0<s<1} \frac{\log M(s,X)}{\log s}, \qquad \bar{\gamma}_X := \inf_{s>1} \frac{\log M(s,X)}{\log s}.$$

For the main properties of the indices see [10]. In particular,

$$0 \leqslant \underline{\gamma}_X \leqslant \bar{\gamma}_X \leqslant 1, \qquad \bar{\gamma}_{X'} = 1 - \underline{\gamma}_X, \qquad \underline{\gamma}_{X'} = 1 - \bar{\gamma}_X, \tag{1.3}$$

the latter following from

$$\tau_X(t)\,\tau_{X'}(t) = t \qquad (t > 0). \tag{1.4}$$

In [10] we showed that these fundamental indices can be interpreted as exponents of a certain submultiplicative function. Since we want to benefit from this fact, we briefly recall the construction:

If g: $(0,\infty) \to (0,\infty)$ is submultiplicative, i.e., $g(st) \leqslant g(s)\,g(t)$ for $s, t > 0$, then the numbers

$$p_0^*(g) := \sup\{p_0 \in \mathbb{R};\, g(s) = \mathcal{O}(s^{p_0}),\ s \to 0+\},$$
$$p_1^*(g) := \inf\{p_1 \in \mathbb{R};\, g(s) = \mathcal{O}(s^{p_1}),\ s \to \infty\}$$

are called the *lower* and *upper exponents* of g, respectively. A particular way of constructing submultiplicative functions was discussed in [12]. If a function ϕ: $(0, \infty) \to (0, \infty)$ has the property $\phi(st) \leqslant c(s)\, \phi(t)$ for $s, t > 0$ with some constant $c(s) > 0$ depending only upon s, then the function g_ϕ, defined by

$$g_\phi(s) := \sup_{t, st \in \Omega^*} \frac{\phi(st)}{\phi(t)} \qquad (s > 0),$$

is submultiplicative. If $c(s) = \max\{s^{p_0}, s^{p_1}\}$, $s > 0$, then ϕ is absolutely continuous and

$$p_0 \leqslant \frac{t\phi'(t)}{\phi(t)} \leqslant p_1 \qquad (t > 0). \tag{1.5}$$

If ϕ is increasing then $p_0^*(g_\phi) \geqslant 0$; moreover in this case $p_1^*(g_\phi) < \infty$ iff $\phi(2t) \leqslant c\phi(t)$, $t \leqslant l/2$, with some constant $c > 0$, i.e., ϕ satisfies the so-called Δ_2-condition for $0 < t \leqslant l/2$. (1.6)

$$p_0^*(g_{\phi^{-1}}) = \frac{1}{p_1^*(g_\phi)}, \qquad p_1^*(g_{\phi^{-1}}) = \frac{1}{p_0^*(g_\phi)}. \tag{1.7}$$

The fundamental indices of the r.i. Banach function space X then result if we choose $\phi = \tau_X$; see [10]. As a corollary we therefore have in particular that τ_X is absolutely continuous, and

$$\underline{\gamma}_X \frac{\tau_X(t)}{t} \leqslant \frac{d\tau_X(t)}{dt} \leqslant \bar{\gamma}_X \frac{\tau_X(t)}{t}. \tag{1.8}$$

The following classes of r.i. spaces are important in connection with growth properties of the fundamental function and the Δ_2-condition:

Let $a \in (0, 1)$, $c > 0$. A r.i. Banach function space $X_\rho(\Omega)$ belongs to the class $\mathscr{U}_{c,a}$ iff there exists a $\delta \in (0, 1)$ such that (1.9)

$$\frac{\tau_X(t_2)}{\tau_X(t_1)} \leqslant c \left(\frac{t_2}{t_1}\right)^a \qquad \left(\frac{t_1}{t_2} \leqslant \delta;\, t_1, t_2 \in \Omega^*\right).$$

Let $b, c > 0$. A r.i. Banach function space $X_\rho(\Omega)$ belongs to the class $\mathscr{L}_{c,b}$ iff there exists a $\delta \in (0, 1)$ such that (1.10)

$$\frac{\tau_X(t_1)}{\tau_X(t_2)} \leqslant c \left(\frac{t_1}{t_2}\right)^b \qquad \left(\frac{t_1}{t_2} \leqslant \delta;\, t_1, t_2 \in \Omega^*\right).$$

These classes $\mathscr{U}_{c,a}$ and $\mathscr{L}_{c,b}$ are refinements of the classes $\mathscr{U}$ and $\mathscr{L}$, respectively, introduced in [27]. Obviously, $\mathscr{U} = \bigcup \{\mathscr{U}_{c,a}; c > 0, a \in (0, 1)\}$ and $\mathscr{L} = \bigcup \{\mathscr{L}_{c,b}; b, c > 0\}$. The following implications hold with respect to the parameters c, a, b of these classes.

LEMMA 1.1. (a) *If* $c > 0$, $0 < a_1 \leqslant a_2 < 1$, *and* $0 < b_1 \leqslant b_2$, *then* $\mathscr{U}_{c,a_1} \subset \mathscr{U}_{c,a_2}$, $\mathscr{L}_{c,b_2} \subset \mathscr{L}_{c,b_1}$;

(b) $X \in \mathscr{U}_{c,a}$ *iff* $X' \in \mathscr{L}_{c,1-a}$.

The inclusions in statement (a) follow directly from (1.9) and (1.10), respectively, whereas statement (b) can be deduced from (1.4). For example, if $X \in \mathscr{U}_{c,a}$, then there exists a number $\delta \in (0, 1)$ such that

$$\frac{\tau_{X'}(t_1)}{\tau_{X'}(t_2)} = \frac{\tau_X(t_2)}{\tau_X(t_1)}, \qquad \frac{t_1}{t_2} \leqslant c \left(\frac{t_1}{t_2}\right)^{1-a}$$

for all $t_1, t_2 \in \Omega^*$ with $t_1/t_2 \leqslant \delta$. This shows already that $X \in \mathscr{L}_{c,1-a}$.

The classes $\mathscr{U}_{c,a}$ and $\mathscr{L}_{c,b}$ are closely related to the Δ_2-condition, and to the fundamental indices as the proposition below shows.

PROPOSITION 1.2. *Assume that* $\tau_X(0+) = 0$, $\tau_X(t) \neq 0$ *for* $t \neq 0$.

(a) (i) $X \in \mathscr{U}_{c,a}$ *iff* $1/\tau_X(1/t)$ *satisfies the* Δ_2*-condition with* $k(s) = cs^a$ *for* $t > 1/l$;

(ii) *if* $X \in \mathscr{U}_{c,a}$, *then* $\bar{\gamma}_X \leqslant a$; *if* $\bar{\gamma}_X < a$, *then* $X \in \mathscr{U}_{1,a}$;

(b) (i) $X \in \mathscr{L}_{c,b}$ *iff* $1/\tau_{X'}(1/t)$ *satisfies the* Δ_2*-condition with* $k(s) = cs^{1-b}$ *for* $t > 1/l$;

(ii) *if* $X \in \mathscr{L}_{c,b}$, *then* $\underline{\gamma}_X \geqslant b$; *if* $\underline{\gamma}_X > b$, *then* $X \in \mathscr{L}_{c,b}$.

Note that the Δ_2-condition for a ϕ-function ϕ for $t > t_0 \geqslant 0$ is equivalent to $\phi(st)/\phi(t) \leqslant k(s)$, $t > t_0$, for any $s > 1$ and some constant $k(s) > 0$, depending only upon s.

The proofs of a(i) and b(i), which use (1.9) and (1.10), are left to the reader. For the proof of a statement a(ii) we assume first that $X \in \mathscr{U}_{c,a}$. On account of (1.9) and (1.2) there exists a $\delta \in (0, 1)$ such that $M(1/s, X) \leqslant cs^{-a}$ for all $s \in (0, \delta)$. For $s > 1/\delta$ we therefore have $M(s, X) \leqslant cs^a$, and hence hat $\bar{\gamma}_X \leqslant a$ by [10, $F1$]. Conversely the assumption $\bar{\gamma}_X < a$ implies on account of [10, $F1$] that there exists a $\delta > 1$ such that $\log M(s, X)/\log s \leqslant a$ for all $s > \delta$, i.e., $M(1/s, X) \leqslant s^{-a}$ for $s < 1/\delta$; this yields $X \in \mathscr{U}_{1,a}$ by (1.9). The proof of statement b(ii) is similar.

COROLLARY 1.3. (a) *The following assertions are equivalent*:

(i) $\bar{\gamma}_X < 1$;

(ii) *$X \in \mathscr{U}_{1,a}$ for some $a \in (0, 1)$;*

(iii) *$1/\tau_X(1/t)$ satisfies the Δ_2-condition with $k(s) = s^a$ for $t > 1/l$;*

(b) *the following assertions are equivalent:*

(i) $\underline{\gamma}_X > 0$;

(ii) *$X \in \mathscr{L}_{1,b}$ for some $b > 0$;*

(iii) *$1/\tau_X(1/t)$ satisfies the Δ_2-condition with $k(s) = s^{1-b}$ for $t > 1/l$.*

2. Geometric Properties of Orlicz and Lorentz Spaces

If $X = L_p$, $1 \leqslant p \leqslant \infty$, is a Lebesgue space, then L_p has an absolutely continuous norm if and only if $p < \infty$; if the measure μ is separable, the space L_p is separable if and only if $p < \infty$; moreover, L_p is uniformly convex if and only if $1 < p < \infty$; finally L_p is reflexive if and only if $1 < p < \infty$. On the other hand, the fundamental function of L_p is given by $\tau_{L_p}(t) = t^{1/p}$, and hence $\underline{\gamma}_{L_p} = \bar{\gamma}_{L_p} = 1/p$. This means that absolute continuity of the norm and separability can both be characterized by the index condition $\underline{\gamma}_{L_p} > 0$, whereas uniform convexity as well as reflexivity of the L_p-space are both equivalent to $0 < \underline{\gamma}_{L_p} \leqslant \bar{\gamma}_{L_p} < 1$. The concrete question now is whether similar statements hold also for Orlicz spaces or Lorentz spaces.

2.1. *The Case of Orlicz Spaces*

First we consider the case when $X = L_{M\Psi}$ is an Orlicz space (for definition of $L_{M\Psi}$, see, e.g. [17, 19, 20]). The surprising fact is that the counterparts of the four results just mentioned for Lebesgue spaces are also valid for Orlicz spaces.

THEOREM 2.1. *Assume that the Young function Ψ is strictly increasing, and let $l = \infty$. Then*

(a) *$L_{M\Psi}$ has an absolutely continuous norm iff $\underline{\gamma}_{L_{M\Psi}} > 0$;*

(b) *if μ is separable, then $L_{M\Psi}$ is separable iff $\underline{\gamma}_{L_{M\Psi}} > 0$;*

(c) *$L_{M\Psi}$ is reflexive iff $0 < \underline{\gamma}_{L_{M\Psi}} \leqslant \bar{\gamma}_{L_{M\Psi}} < 1$;*

(d) *$L_{M\Psi}$ is uniformly convex iff Ψ is uniformly convex and $\underline{\gamma}_{L_{M\Psi}} > 0$.*

Note that a Young function Ψ is called uniformly convex if and only if for each $a \in (0, 1)$ there exists a $\delta \in (0, 1)$ such that (see [16])

$$\Psi\left(\frac{t + ht}{2}\right) \leqslant (1 - \delta)\,\frac{\Psi(t) + \Psi(th)}{2} \qquad (t \geqslant 0, h \in [0, a]).$$

The crucial point of the proof of Theorem 2.1 is the following

PROPOSITION 2.2. *Under the assumption of Theorem* 2.1 *the Young function Ψ satisfies the Δ_2-condition for $t > 0$ iff $\underline{\gamma}_{L_{M\Psi}} > 0$.*

Proof of Proposition 2.2. First assume that $\underline{\gamma}_{L_{M\Psi}} > 0$. By Corollary 1.3 there exists a number $b > 0$ such that $L_{M\Psi} \in \mathscr{L}_{1,b}$. In order to apply Proposition 1.2(b) recall that the fundamental function of $L_{M\Psi}$ is given by $\tau_{L_{M\Psi}}(t) = 1/\Psi^{-1}(1/t)$, $t > 0$, (see, e.g. [4, 7, 23]). Because of (1.4) one has $\tau'_{L_{M\Psi}}(t) = t\Psi^{-1}(1/t)$, $t > 0$. So Proposition 1.2(b) yields in the Orlicz case that $L_{M\Psi} \in \mathscr{L}'_{1,b}$ for some $b > 0$ if and only if for some $s > 1$

$$s\Psi^{-1}(t)/\Psi^{-1}(st) \leqslant s^{1-b} \qquad (t > 0).$$

This means that $1/[\Psi^{-1}(t)/\Psi^{-1}(st)] \geqslant s^b$ for $t > 0$. Passing to the supremum, we obtain

$$s^b \leqslant \frac{1}{\sup_{t>0}[\Psi^{-1}(t)/\Psi^{-1}(st)]} = \left[\sup_{t>0} \frac{\Psi(\cdot\, t)}{\Psi(t)}\right]^{-1}(s)$$

on account of the strict monotonicity of Ψ (see [4]). This is equivalent to $\sup_{t>0} \Psi(s^b t)/\Psi(t) \leqslant s$. Since $s > 1$ implies $s^b > 1$, this is the Δ_2-condition for Ψ.

Conversely, assume that Ψ satisfies the Δ_2-condition. The submultiplicative function g_Ψ then has a finite upper exponent because of (1.6). Therefore and on behalf of (1.7) the lower exponent of $g_{\Psi^{-1}}$ is positive, i.e., $p_0^*(g_{\Psi^{-1}}) > 0$. On the other hand, we showed in [9, 10] that

$$\underline{\gamma}_{L_{M\Psi}} = p_0^*(g_{\tau_X}) = p_0^*(g_{1/\Psi^{-1}(1/t)}) = -p_1^*(g_{\Psi^{-1}(1/t)}) = p_0^*(g_{\Psi^{-1}}).$$

In other words, $\underline{\gamma}_{L_{M\Psi}} > 0$.

Proof of Theorem 2.1. The Orlicz space $L_{M\Psi}$ has an absolutely continuous norm if and only if (see [20, p. 58, 59]) the Young function Ψ satisfies the Δ_2-condition for all $t > 0$. Because of Proposition 2.2 this is equivalent to $\underline{\gamma}_{L_{M\Psi}} > 0$, yielding statement (a). Statement (b) follows from (a) since any r.i. Banach function space with a separable measure μ is separable if and only if the norm of the space is absolutely continuous. Moreover, a r.i. Banach function space is reflexive, if and only if both the norms of the space itself as well as the norm of its associate space are absolutely continuous. Because of part (a) the Orlicz space $L_{M\Psi}$ is reflexive if and only if both $\underline{\gamma}_{L_{M\Psi}} > 0$ and $\underline{\gamma}_{L'_{M\Psi}} > 0$. By (1.3) the latter condition is equivalent to $\underline{\gamma}_{L_{M\Psi}} < 1$; so statement (c) follows if we recall that $\underline{\gamma}_{L_{M\Psi}} \leqslant \bar{\gamma}_{L_{M\Psi}}$ by (1.3). Finally, assertion (d) follows from Proposition 2.2 by means of a theorem of [16] stating that $L_{M\Psi}$ is uniformly convex iff Ψ is uniformly convex and satisfies the Δ_2-condition for all $t > 0$.

Remark. According to the famous theorem of Milman, any uniform convex Banach space is reflexive. If, in particular, the Banach space is an Orlicz space then this statement follows directly from Theorem 2.1(c) and (d). Indeed, if $L_{M\Psi}$ is uniformly convex, then $\underline{\gamma}_{L_{M\Psi}} > 0$, and we only have to show that the uniform convexity of the Young function Ψ implies the second half of the index condition in statement (c), namely, $\bar{\gamma}_{L_{M\Psi}} < 1$. For this purpose, observe that the uniform convexity of Ψ implies hat the Young function Φ satisfies the Δ_2-condition. Therefore $\underline{\gamma}_{(L_{M\Psi})'} > 0$ by Proposition 2.2 and on account of the fact that the conjugate Young function Φ generates the associate space $(L_{M\Psi})'$. By (1.3) it follows that $\bar{\gamma}_{L_{M\Psi}} = 1 - \underline{\gamma}_{(L_{M\Psi})'} < 1$, i.e., $\bar{\gamma}_{L_{M\Psi}} < 1$ as desired. On the other hand, Proposition 2.2 shows that the converse of Milman's theorem (i.e., reflexcivity implies uniform convexity of the space), which is valid for Lebesgue spaces but not generally true, is also not true for Orlicz spaces. In fact, Milnes [24] gives an example of a Young function Ψ whose conjugate function Φ satisfies the Δ_2-condition (and hence $\bar{\gamma}_{L_{M\Psi}} < 1$ by Proposition 2.2), but Ψ is not uniformly convex. Hence the second half of the index condition in statement (c) does not imply the uniform convexity of Ψ which would be necessary for the uniform convexity of the space $L_{M\Psi}$.

2.2 *Properties of Lorentz Spaces*

Our next example is the case when $X = \Lambda(\phi, p)$, $1 < p < \infty$, is a generalized Lorentz space (for definition, see, e.g. [25, 27, 30]).

THEOREM 2.3. *Assume that* $\phi(t) \neq 0$ *for* $t \neq 0$ *and* $l = \infty$.

(a) *If* $\underline{\gamma}_{\Lambda(\phi,p)} > 0$, *then* $\Lambda(\phi, p)$ *is reflexive*;

(b) $\Lambda(\phi, p)$ *is uniformly convex iff* $\underline{\gamma}_{\Lambda(\phi,p)} > 0$.

Remark. For $l < \infty$, $1 < p < \infty$ the space $\Lambda(\phi, p)$ is always reflexive [13].

For the proof of this theorem note that the fundamental function of the Lorentz space $\Lambda(\phi, p)$ is equal to

$$\tau_{\Lambda(\phi,p)}(t) = \left(\int_0^t \phi(u)\,du\right)^{1/p} =: \Phi(t)^{1/p}, \tag{2.1}$$

and $\tau_{\Lambda(\phi,p)'}(t) = t(\Phi(t))^{-1/p}$ by (1.4).

First we prove statement (a). If $\underline{\gamma}_{\Lambda(\phi,p)} > 0$, then $\Lambda(\phi, p) \in \mathscr{L}_{1,b}$ for some $b > 0$ because of Corollary 1.3. Therefore we have by Proposition 1.2 that for any $s > 1$

$$s\,\frac{\Phi(st)^{-1/p}}{\Phi(t)^{-1/p}} \leqslant s^{1-b} \qquad (t > 0),$$

i.e., $s^{bp}\Phi(t) \leqslant \Phi(st)$. Fixing $t > 0$ and letting s tend to infinity shows that

$$\lim_{s\to\infty} \int_0^s \phi(u)\, du \equiv \lim_{s\to\infty} \Phi(s) = \infty.$$

Hence $\phi \notin L_1(0, \infty)$, and by a result of Halperin [15] (see also [2]) this is equivalent to the reflexivity of the Lorentz space $\Lambda(\phi, p)$.

As mentioned before, for statement (b) we use the theorem of Halperin [14] stating that $\Lambda(\phi, p)$, $p > 1$, is uniformly convex if and only if $N(s) < 1$ for some $s > 1$, where

$$N(s) := \sup_{t>0} \frac{\Phi(t)}{\Phi(st)}.$$

First assume that $\underline{\gamma}_{\Lambda(\phi,p)} > 0$ or, equivalently, that $\Lambda(\phi, p) \in \mathscr{L}_{1,b}$ for some $b > 0$. As above it follows for any $s > 1$ that $N(s) \leqslant s^{-bp}$, and therefore $N(s) < 1$ for $s > 1$, i.e., the space $\Lambda(\phi, p)$ is uniformly convex. Conversely, if $\Lambda(\phi, p)$ is uniformly convex, then there exists a number $s_0 > 1$ such that $N(s_0) < 1$. So we can conclude by [2] that $N(s) < 1$ for all $s > 1$. On the other hand,

$$N(s) = M(1/s, \Lambda(\phi, p))^{1/p} \qquad (s > 1)$$

because of (1.2) and (2.1). Hence the upper exponent of the submultiplicative function $M(1/\cdot, \Lambda(\phi, p))^{1/p}$ is negative, i.e.,

$$p_1^*(M(1/\cdot, \Lambda(\phi, p)))^{1/p} < 0.$$

Since the lower fundamental index of $\Lambda(\phi, p)$ is equal to the lower exponent of the submultiplicative function $M(\cdot, \Lambda(\phi, p))$ (as proved in [10]), this implies that

$$\begin{aligned}\underline{\gamma}_{\Lambda(\phi,p)} = p_0^*(M(\cdot, \Lambda(\phi, p))) &= \lim_{s\to 0+} \frac{\log M(s, \Lambda(\phi, p))}{\log s} \\ &= -\lim_{s\to\infty} \frac{\log M(1/s, \Lambda(\phi, p))}{\log s} = -p_1^*(M(1/\cdot, \Lambda(\phi, p))) > 0,\end{aligned}$$

i.e., $\underline{\gamma}_{\Lambda(\phi,p)} > 0$, as asserted.

Remark. When comparing Theorem 2.3 with Theorem 2.1 note that the norm of the Lorentz space $\Lambda(\phi, p)$, $p < \infty$, is always absolutely continuous and so $\Lambda(\phi, p)$ is always separable provided the measure μ is separable. Hence the counterparts of Theorem 2.1(a) and (b) for Lorentz spaces are obvious.

All in all, the latter two relsults on Lebesgue spaces stated in the introduction of this subsection carry over to Lorentz spaces. The first two are always true.

3. Boyd Indices and Spaces of Fundamental Type

The Boyd indices [3] of a r.i. Banach function space $X \equiv X_\rho(\Omega)$, defined by means of the indicator function of X as

$$\alpha_X := \inf_{0<s<1} -\frac{\log h(s, X)}{\log s}, \qquad \beta_X := \sup_{s>1} -\frac{\log h(s, X)}{\log s} \tag{3.1}$$

have properties similar to those of the fundamental indices. In particular,

$$0 \leqslant \beta_X \leqslant \alpha_X \leqslant 1, \qquad \alpha_{X'} = 1 - \beta_X, \qquad \beta_{X'} = 1 - \alpha_X, \tag{3.2}$$

where the latter is a consequence of

$$sh(s, X') = h(1/s, X) \qquad (s > 0). \tag{3.3}$$

In the Lebesgue case $X = L_p$, $1 \leqslant p \leqslant \infty$, we have $h(s, L_p) = s^{-1/p}$ and $\alpha_{L_p} = \beta_{L_p} = 1/p$, i.e., the Boyd indices for Lebesgue spaces coincide with their fundamental indices. Generally, it can be proved that

$$0 \leqslant \beta_X \leqslant \underline{\gamma}_X \leqslant \bar{\gamma}_X \leqslant \alpha_X \leqslant 1. \tag{3.4}$$

Indeed, by (1.2) we have for any $s > 0$

$$M(s, X) = \sup_{t, st \in \Omega^*} \frac{\|\chi_{0,st)}\|_{X(\Omega^*)}}{\|\chi_{(0,t)}\|_{X(\Omega^*)}} = \sup_{t, st \in \Omega^*} \frac{\|E_{1/s}\chi_{(0,t)}\|_{X(\Omega^*)}}{\|\chi_{(0,t)}\|_{X(\Omega^*)}};$$

hence $M(s, X) \leqslant h(1/s, X)$, $s > 0$. Inserting this estimate into Definition 1.1 and (3.1), respectively, now yields (3.4). Moreover, this calculation shows that the auxiliary function $M(\cdot, X)$, up till now of mere technical meaning, can be interpreted as the "norm" of the restriction of the dilation operator $E_{1/s}$ to the set of characteristic functions of intervals $(0, t)$. This new insight gives rise to the following definition which selects out an important subclass of r.i. spaces:

Definition 3.1. A r.i. Banach function space X is said to be

(a) of upper (lower) fundamental type iff $\alpha_X = \bar{\gamma}_X (\beta_X = \underline{\gamma}_X)$;

(b) of fundamental type iff $\alpha_X = \bar{\gamma}_X$ and $\beta_X = \underline{\gamma}_X$.

Obviously, any space X with $M(s, X) = h(1/s, X)$, $s > 0$, is of fundamental type. Further, in [7] we proved the following:

THEOREM 3.1. *If X is a Lebesgue space L_p, $1 \leqslant p \leqslant \infty$, or a Lorentz space L_{pq}, $1 \leqslant p, q < \infty$, or a generalized Lorentz space $\Lambda(\phi, p)$, $1 \leqslant p < \infty$, or an Orlicz space $L_{M\Psi}$ with strictly increasing Young function Ψ, then $M(s, X) = h(1/s, X)$, $s > 0$, and hence X is of fundamental type.*

The next theorem gives necessary and sufficient conditions for a function space to be of fundamental type. These conditions are formulated in a concrete form in order to have a criterium which can be used practically, to test whether some r.i. space is of fundamental type or not.

THEOREM 3.2. *Let X be any r.i. Banach function space.*

(a) *The following statements are equivalent*:

(a1) *X is of upper fundamental type*;

(a2) *for each $\varepsilon > 0$ there exists a number $\delta \in (0, 1)$ such that*

$$s^{\varepsilon} \leqslant M(1/s, X)/h(s, X) \leqslant s^{-\varepsilon} \qquad (0 < s \leqslant \delta);$$

(a3) *for each $\varepsilon > 0$, $\int_0^1 [M(1/s, X)/h(s, X)]\, ds/s^{1+\varepsilon} = \infty$*;

(a4) *for each $\varepsilon > 0$, $\int_0^1 [h(s, X)/M(1/s, X)]\, ds/s^{1-\varepsilon} < \infty$.*

(b) *The following statements are equivalent*:

(b1) *X is of lower fundamental type*;

(b2) *for each $\varepsilon > 0$ there exists a number $R > 1$ such that*

$$s^{-\varepsilon} \leqslant M(1/s, X)/h(s, X) \leqslant s^{\varepsilon} \qquad (s \geqslant R);$$

(b3) *for each $\varepsilon > 0$, $\int_1^{\infty} M(1/s, X)\, h(s, X)]\, ds/s^{1-\varepsilon} = \infty$*;

(b4) *for each $\varepsilon > 0$, $\int_1^{\infty} [h(s, X)/M(1/s, X)]\, ds/s^{1+\varepsilon} < \infty$.*

The proof of (a) will be organized as follows: First we show that statement (a1) implies (a2), (a2) implies (a3), and (a3) implies (a1), furnishing the equivalence of statements (a1)–(a3). Then we show that statement (a2) implies (a4), and (a4) implies (a1).

For the implication (a1) $\Rightarrow$ (a2) note that for each $\varepsilon > 0$ there exist numbers $\delta_1, \delta_2 \in (0, 1)$ such that $s^{-\bar{\gamma}_X} \leqslant M(1/s, X) \leqslant s^{-\bar{\gamma}_X - \varepsilon}$ for $0 < s \leqslant \delta_1$, and $s^{-\alpha_X} \leqslant h(s, X) \leqslant s^{-\alpha_X - \varepsilon}$ for $0 < s \leqslant \delta_2$; see [10]. With $\delta := \min\{\delta_1, \delta_2\}$ we deduce

$$s^{\alpha_X - \bar{\gamma}_X + \varepsilon} \leqslant \frac{M(1/s, X)}{h(s, X)} \leqslant s^{\alpha_X - \gamma_X - \varepsilon} \qquad (0 < s \leqslant \delta), \tag{3.5}$$

and hence (a2), since $\alpha_X = \bar{\gamma}_X$ by assumption (a1). If (a2) is valid, then for each $\varepsilon > 0$ and some suitable $\delta \in (0, 1)$

$$\int_0^1 \frac{M(1/s, X)}{h(s, X)} \frac{ds}{s^{1+\varepsilon}} \geqslant \int_0^\delta \frac{M(1/s, X)}{h(s, X)} \frac{ds}{s^{1+\varepsilon}} \geqslant \int_0^\delta \frac{ds}{s} = \infty,$$

and we have statement (a3). Now, suppose the integral in statement (a3) diverges for each $\varepsilon > 0$, and let $\delta \in (0, 1)$ be as in (3.5). Since $M(1/s, X) \leqslant h(s, X)$,

$$\int_\delta^1 \frac{M(1/s, X)}{h(s. X)} \frac{ds}{s^{1+\varepsilon}} \leqslant \int_\delta^1 \frac{ds}{s^{1+\varepsilon}} < \infty.$$

Because of statement (a3) this implies the divergence of the integral $\int_0^\delta [M(1/s, X)/h(s, X)] \, ds/s^{1+\varepsilon}$. A fortiori and by (3.5) we therefore have that $\int_0^\infty s^{\alpha_X - \bar{\gamma}_X} \, ds/s^{1+\varepsilon} = \infty$. This is only possible if there exists a number $s_\varepsilon \in (0, \delta)$, depending on ε, such that $s_\varepsilon^{\alpha_X - \bar{\gamma}_X - 1 - 2\varepsilon} \geqslant s_\varepsilon^{\varepsilon - 1}$, i.e., $(\alpha_X - \bar{\gamma}_X) \log s_\varepsilon \geqslant 2\varepsilon \log s_\varepsilon$, or, equivalently, $\alpha_X - \bar{\gamma}_X \leqslant 3\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain statement (a1). If statement (a2) is assumed, then for each $\varepsilon > 0$ there exists a $\delta \in (0, 1)$ such that the critical integral in (a4) is finite, namely,

$$\int_0^\delta \frac{h(s, X)}{M(1/s, X)} \frac{ds}{s^{1-\varepsilon}} \leqslant \int_0^\delta s^{\varepsilon/2 - 1} \, ds < \infty.$$

Moreover, it is known [3, 10] that $h(s, X) \leqslant \max\{1, 1/s\}$ and $M(1/s, X) \leqslant \min\{1, 1/s\}$, $s > 0$. Hence,

$$\int_0^1 \frac{h(s, X)}{M(1/s, X)} \frac{ds}{s^{1-\varepsilon}} \leqslant \int_0^1 \frac{ds}{s^{2-\varepsilon}} < \infty,$$

yielding statement (a4). Finally, if statement (a4) is valid, then a fortiori $\int_0^\delta s^{\alpha_X - \bar{\gamma}_X + \varepsilon} \, ds/s^{1-\varepsilon} < \infty$, because of (3.5). Analogously to our above considerations we can conclude that there exists a number $s_\varepsilon \in (0, \delta)$ such that $s_\varepsilon^{-\alpha_X + \bar{\gamma}_X + 2\varepsilon - 1} < s_\varepsilon^{-1}$, i.e., $\alpha_X - \bar{\gamma}_X < 2\varepsilon$. This implies statement (a1).

The equivalence of the statements in (b) follows from (a) by duality arguments, in particular by replacing X by X' and observing that $X'' = X$ as well as (1.3), (3.2), (1.4), and (3.3). The detailed proof is left to the reader.

As a first application of Theorem 3.2 we show that the function space S, which was introduced by Shimogaki in [29], is not of fundamental type, and hence Definition 3.1 makes sense. The Shimogaki space S is defined as

$$S := \{f \in \mathscr{M}([0, 1]) : \|f\|_S < \infty\}$$

with norm $\|f\|_S := \sup_{\varphi \in C} \int_0^1 \varphi(t) f^*(t)\, dt$, where f^* again denotes the nonincreasing rearrangement of f, and C a certain class of functions; see [29] for details. A simple calculation shows that $\tau_S(t) = t^{1/2}$. Therefore, $M(s, S) = s^{1/2}$ and $\underline{\gamma}_S = \bar{\gamma}_S = 1/2$. On the other hand, it is very difficult to compute the indicator function of S. However, it can be shown that $\lim_{s \to \infty} h(s, S) \geqslant 1$, and this is sufficient for our purpose. Indeed, on account of the monotonicity it follows that $h(s, S) \geqslant 1$ for $s > R$ and some $R > 1$. Hence $\int_1^\infty [h(s, S)/M(1/s, S)]\, ds/s^{1+\varepsilon} \geqslant \int_R^\infty s^{1/2}\, ds/s^{1+\varepsilon} = \infty$ for each $\varepsilon \in (0, 1/2)$, and Theorem 3.2 yields

COROLLARY 3.3. *The Shimogaki space S is not of fundamental type.*

Of more theoretical interest than Theorem 3.2 is the following theorem which gives a sufficient condition for spaces to be of fundamental type by means of the classes $\mathscr{U}_{c,a}$, $\mathscr{L}_{c,b}$, that is to say, by means of growth conditions upon the fundamental function and Boyd indices.

THEOREM 3.4. *If $X \in \mathscr{L}_{c,b}$ with $b \geqslant \alpha_X$, then X is of upper fundamental type. If $X \in \mathscr{U}_{c,a}$ with $0 < a \leqslant \beta_X$, then X is of lower fundamental type.*

For the proof assume $X \in \mathscr{L}_{c,b}$ with $b \geqslant \alpha_X$, and let $\varepsilon > 0$ and $b_1 := \alpha_X + \varepsilon$. Then there exists a $\delta \in (0, 1)$ such that for all $s \in (0, \delta]$

$$h(s, X) \leqslant s^{-b_1} \leqslant c \frac{\tau_X(t)}{\tau_X(st)} s^{b-b_1} \leqslant cM(1/s, X)\, s^{b-\alpha_X-\varepsilon},$$

and the integral in (a3) can be estimated by

$$\int_0^1 \frac{M(1/s, X)}{h(s, X)} \frac{ds}{s^{1+\varepsilon}} \geqslant \frac{1}{c} \int_0^\delta s^{-b+\alpha_X+\varepsilon} \frac{ds}{s^{1+\varepsilon}} = \infty$$

since $\alpha_X \leqslant b$. So X is of upper fundamental type on account of Theorem 3.2. The second statement in Theorem 3.4 is obtained by duality arguments, observing Lemma 1.1(b).

Summing up, one can say that the class of r.i. spaces of fundamental type is a proper subclass of all r.i. spaces, but it still contains most of the spaces which are important for applications.

4. PARTICULAR SPACES OF FUNDAMENTAL TYPE

The aim of this section is to assign to each r.i. Banach function space $X \equiv X_\rho(\Omega)$ with $\tau_X(0+) = 0$ and τ_X concave (this assumption can be made without loss of generality, see, e.g. [30]) in a canonical way two r.i. Banach function spaces which have the same fundamental function as X, but are of fundamental type. These are the Lorentz space $\mathbf{\Lambda}(X)$, $\mathbf{M}(X)$ associated with X and defined by

$$\mathbf{\Lambda}(X) := \left\{ f \in \mathscr{M}(\Omega); \|f\|_{\mathbf{\Lambda}(X)} := \int_0^l f^*(s)\, d\tau_X(s) < \infty \right\}, \tag{4.1}$$

$$\mathbf{M}(X) := \left\{ f \in \mathscr{M}(\Omega); \|f\|_{\mathbf{M}(X)} := \sup_{t \in \Omega^*} (\tau_X(t)/t) \int_0^t f^*(s)\, ds < \infty \right\}. \tag{4.2}$$

It is well known that $\mathbf{\Lambda}(X)$, $\mathbf{M}(X)$ are r.i. Banach function spaces such that $\mathbf{\Lambda}(X) \subset X \subset \mathbf{M}(X)$ with continuous embeddings, and

$$\mathbf{\Lambda}(X)^* = \mathbf{\Lambda}(X)' = \mathbf{M}(X'); \qquad \mathbf{\Lambda}(X') = \mathbf{M}^0(X)^* = \mathbf{M}^0(X)', \tag{4.3}$$

where $\mathbf{M}^0(X)$ denotes the norm closure of the set of simple functions in $\mathbf{M}(X)$ with finite support, and * indicates the dual of a Banach space, see [28, 25, 30]. We now prove that the spaces $\mathbf{\Lambda}(X)$ and $\mathbf{M}(X)$, which play an important role in interpolation theory [30, 27, 11], are of fundamental type. More precisely

THEOREM 4.1. (a) $\tau_{\mathbf{\Lambda}(X)} = \tau_{\mathbf{M}(X)} = \tau_X$;

(b) *both* $\mathbf{\Lambda}(X)$ *and* $\mathbf{M}(X)$ *are of fundamental type with* $h(s, \mathbf{\Lambda}(X)) = h(s, \mathbf{M}(X)) = M(1/s, X)$.

The proof of statement (a) is straightforward: For each $t \in (0, l]$ and $E \in \Sigma$ with $\mu(E) = t$ we have

$$\tau_{\mathbf{\Lambda}(X)}(t) = \|\chi_E\|_{\mathbf{\Lambda}(X)} = \int_0^l \chi_E^*(s)\, d\tau_X(s) = \tau_X(t),$$

since $\chi_E^*(t) = \chi_{(0,t)}^*$ and $\tau_X(0+) = 0$ by assumption. For $t > l$ we obtain $\tau_{\mathbf{\Lambda}(X)}(t) = \tau_{\mathbf{\Lambda}(X)}(l) = \tau_X(l) = \tau_X(t)$. Combining both results yields $\tau_{\mathbf{\Lambda}}(X) = \tau_X$. With respect to $\mathbf{M}(X)$ we have for $t \in (0, l]$

$$\tau_{\mathbf{M}(X)}(t) = \|\chi_E\|_{\mathbf{M}(X)} = \sup_{s \in \Omega^*} \frac{\tau_X(s)}{s} \int_0^s \chi_E^*(u)\, du$$

$$= \sup_{s \in \Omega^*} \left\{ \frac{\tau_X(s)}{s} \min\{t, s\} \right\}.$$

Since $(\tau_X(s)/s) \min\{t, s\} = \tau_X(s)$ if $s \leqslant t \leqslant l$, and $(\tau_X(s)/s) \min\{t, s\} = \tau_X(s)\, t/s \leqslant \tau_X(s)$ if $t < s \leqslant l$, it follows that $\tau_{\mathbf{M}(X)}(t) = \sup_{0 < s \leqslant t} \tau_X(s) = \tau_X(t)$ because of the monotonicity of τ_X. For $t > l$ we finally deduce $\tau_{\mathbf{M}(X)}(t) = \tau_{\mathbf{M}(X)}(l) = \tau_X(l) = \tau_X(t)$.

For the proof of statement (b) we begin with the space $\mathbf{M}(X)$. Since $h(s, \mathbf{M}(X)) \geqslant M(1/s, X)$ (see Section 3) it suffices to estimate $h(s, \mathbf{M}(X))$, $s > 0$ from above: For any $f \in \mathbf{M}(X)$ we have

$$\begin{aligned} \|E_s f\|_{\mathbf{M}(X)} &= \sup_{t \in \Omega^*} \left[\frac{\tau_X(t)}{st} \int_0^{st} f^*(v)\,dv\right] \\ &= \sup_{t, st \in \Omega^*} \left[\frac{\tau_X(t)}{st} \int_0^{st} f^*(v)\,dv\right] \\ &= \sup_{t, st \in \Omega^*} \left[\frac{\tau_X(t)}{\tau_X(st)} \frac{\tau_X(st)}{st} \int_0^{st} f^*(v)\,dv\right] \\ &\leqslant M(1/s, X)\,\|f\|_{\mathbf{M}(X)}, \end{aligned}$$

and hence $h(s, \mathbf{M}(X)) \leqslant M(1/s, \mathbf{M}(X))$ on account of statement (a). The statement for $\mathbf{\Lambda}(X)$, namely, $h(s, \mathbf{\Lambda}(X)) = h(s, X)$, follows by duality arguments: Indeed, using (3.3), (4.3), and (1.4),

$$\begin{aligned} h(s, \mathbf{\Lambda}(X)) &= \frac{1}{s} h(1/s, \mathbf{\Lambda}(X)') = \frac{1}{s} h(1/s, \mathbf{M}(X')) = \frac{1}{s} M(s, \mathbf{M}(X')) \\ &= \frac{1}{s} \sup_{t, st \in \Omega^*} \frac{\tau_{X'}(st)}{\tau_{X'}(t)} = \sup_{t, st \in \Omega^*} \frac{\tau_X(t)}{\tau_X(st)}. \end{aligned}$$

As a follow-up of Theorem 4.1 we now have

Proposition 4.2. (a) $\mathbf{\Lambda}(X)$ ($\mathbf{M}(X)$) *is the smallest* (*largest*) *r.i. Banach function space contained in* (*containing*) X *with the same fundamental function*;

(b) $\mathbf{\Lambda}(\mathbf{\Lambda}(X)) = \mathbf{\Lambda}(\mathbf{M}(X)) = \mathbf{\Lambda}(X)$, $\mathbf{M}(\mathbf{\Lambda}(X)) = \mathbf{M}(\mathbf{M}(X)) = \mathbf{M}(X)$;

(c) (i) $X \in \mathscr{U}_{c,a} \Leftrightarrow \mathbf{\Lambda}(X) \in \mathscr{U}_{c,a} \Leftrightarrow \mathbf{M}(X) \in \mathscr{U}_{c,a}$;

(ii) $X \in \mathscr{L}_{c,b} \Leftrightarrow \mathbf{\Lambda}(X) \in \mathscr{L}_{c,b} \Leftrightarrow \mathbf{M}(X) \in \mathscr{L}_{c,b}$;

(d) (i) $X \in \mathscr{U} \Leftrightarrow \beta_{\mathbf{\Lambda}(X)} > 0 \Leftrightarrow \beta_{\mathbf{M}(X)} > 0$;

(ii) $X \in \mathscr{L} \Leftrightarrow \alpha_{\mathbf{\Lambda}(X)} > 0 \Leftrightarrow \alpha_{\mathbf{M}(X)} > 0$;

(e) (i) *if* $X \in \mathscr{U}$, *then* $c_X := \int_0^1 h(s, \mathbf{\Lambda}(X))\,ds < \infty$ *and*

$$\int_0^t \frac{ds}{\tau_X(s)} \leqslant c_X \tau_{X'}(t) \qquad (0 < t < l); \tag{4.4}$$

(ii) *if* $X \in \mathscr{L}$, *then* $c_{X'} := \int_0^1 h(s, \mathbf{\Lambda}(X'))\,ds < \infty$ *and*

$$t \int_t^l \frac{ds}{s\tau_X(s)} \leqslant c_{X'} \tau_{X'}(t) \qquad (0 < t < l). \tag{4.5}$$

Proof. On the one hand, $\tau_{\Lambda(X)} = \tau_X = \tau_{\mathbf{M}(X)}$ and $\Lambda(X) \subset X \subset \mathbf{M}(X)$. On the other hand, if $Y \subset X$ is any r.i. Banach function space with $\tau_Y = \tau_X$, then $\Lambda(X) = \Lambda(Y) \subset Y \subset X$, i.e., $\Lambda(X) \subset Y$. Analogously, if $Z \supset X$ is any r.i. Banach function space with $\tau_Z = \tau_X$, then $\mathbf{M}(X) = \mathbf{M}(Z) \supset Z \supset X$, i.e., $\mathbf{M}(X) \supset Z$, proving assertion (a). Assertions (b) and (c) follow from Theorem 4.1(a), whereas assertion (d) can be deduced from Theorem 4.1(b) and Corollary 1.3. The inequalities (4.4), (4.5) are based on the fact that $\alpha_{\Lambda(X)} < 1$ if and only if $c_X < \infty$, and $\beta_{\Lambda(X)} > 0$ if and only if $c_{X'} < \infty$, respectively (see [2, 5]). On account of Theorem 4.1(b) we have $h(s, \Lambda(X)) \geqslant \tau_X(t)/\tau_X(st)$ for $0 < t < l$, and therefore noting (1.4),

$$c_X \geqslant \tau_X(t) \int_0^1 \frac{ds}{\tau_X(st)} = \frac{1}{\tau_{X'}(t)} \int_0^t \frac{du}{\tau_X(u)},$$

$$c_{X'} \geqslant \tau_X(t) \int_0^{l/t} \frac{ds}{s\tau_X(st)} = \frac{t}{\tau_{X'}(t)} \int_t^l \frac{du}{u\tau_X(u)}.$$

In particular, part (a) shows that the spaces $\Lambda(X)$ and $\mathbf{M}(X)$, assigned to X, are in a certain sense optimal (for this matter, see also [11]), whereas (4.5) improves a result of [27] which is essential in interpolation theory [11]. The dual version of (4.5) is (4.6). The value of the constants c_X and $c_{X'}$ stems from a generalized Hardy inequality [5].

Finally let us note some examples. We have $\Lambda(L_p) = \Lambda(L_{pq}) = L_{p1}$, $\mathbf{M}(L_p) = \mathbf{M}(L_{pq}) = L_{p\infty}$ for $1 \leqslant p < \infty$; $\Lambda(\Lambda(\phi, p)) = \Lambda(\Phi^{1/p-1}\phi, 1)$ for $1 \leqslant p < \infty$; $\Lambda(L_{M\Psi}) = \Lambda(1/[s\Psi^{-1}(1/s)]^2\, \Psi'(1/s), 1)$ if Ψ is strictly increasing and $\Lambda(S) = L_{21}$, $\mathbf{M}(S) = L_{2\infty}$. These examples, in particular the example of the Shimogaki space S, show that the process of passing from a r.i. Banach function space X to the extremal space of fundamental type is a smoothing effect.

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References

1. B. A. Akamovič, On uniformly convex and uniformly smooth Orlicz spaces, *Teor. Funkciĭ Funkcional. Anal. i Priložen* **15**(1972), 114–220. [Russian]
2. D. W. Boyd, The Hilbert transform on rearrangement invariant spaces, *Canad. J. Math.* **19**((1967), 599–616.

3. D. W. BOYD, Indices of function spaces and their relationship to interpolation. *Canad. J. Math.* **21**(1969), 1245–1254.
4. D. W. BOYD, Indices for Orlicz spaces, *Pacific J. Math.* **38** (1971), 315–323.
5. P. L. BUTZER AND F. FEHÉR, Generalized Hardy and Hardy–Littlewood inequalities in rearrangement-invariant spaces, *Comment. Math. Pace Mat.* Tomus Specialis in Honorem Ladislai Orlicz **I** ((1978), 41–64.
6. F. FEHÉR, A note on a paper of E. R. Love, *Bull. Austral. Math. Soc.* **19** (1978), 67–75.
7. F. FEHÉR, A generalized Schur–Hardy inequality in normed Köthe spaces, *in* "General Inequalities II" (E. F. Beckenbach, Ed.), pp. 277–286, Birkhäuser, Basel, 1980.
8. F. FEHÉR, A note on weak-type interpolation and function spaces, *Bull. London. Math. Soc.* **12** (1980), 443–451.
9. F. FEHÉR, "Interpolation und Indices in symmetrischen Funktionenräumen," Habilitationsschift, Aachen Technical University, 1981.
10. F. FEHÉR, Exponents of submultiplicative functions and function spaces, *in* "General Inequalities III" (E. F. Beckenbach, Ed.), Birkhäuser, Basel, in press.
11. F. FEHÉR, The Marcinkievicz interpolation theorem for rearrangement-invariant function spaces and applications, in preparation.
12. J. GUSTAVSSON AND J. PEETRE, Interpolation of Orlicz spaces, *Studia Math.* **50** (1977), 33–59.
13. I. HALPERIN, Function spaces, *Canad. J. Math.* **5** (1953), 273–288.
14. I. HALPERIN, Uniform convexity in function spaces, *Duke. Math. J.* **21**, (1954), 195–204.
15. I. HALPERIN, Reflexivity in the L^{λ} function spaces. *Duke. Math. J.* **21** (1954), 205–208.
16. A. KAMIŃSKA, On uniform convexity of Orlicz spaces, *Indag. Math.* in press.
17. M. A. KRASNOSELSKII AND YA. B. RUTICKII, "Convex Functions and Orlicz Spaces," Noordhoff, Groningen, 1961.
18. J. LINDENSTRAUSS AND L. TZAFRIRI, "Classical Banach Spaces II. Function Spaces," Springer-Verlag, Berlin/Heidelberg/New York, 1979.
19. G. G. LORENTZ, "Bernstein Polynomials," Univ. of Toronto Press, Toronto, 1953.
20. W. A. J. LUXEMBURG, "Banach Function Spaces," Thesis, Delft Technical University, 1955.
21. W. A. J. LUXEMBURG, "Rearrangement-Invariant Banach Function Spaces," Queen's Papers in Pure and Applied Mathematics, Vol. 10, pp. 83–144, Queen's University, Canada, 1967.
22. W. A. J. LUXEMBURG AND A. C. ZAANEN, "Riesz Spaces I," North-Holland, Amsterdam, 1971.
23. M. MILMAN, "Some New Function Spaces and Their Tensor Products," Notas de Mathematica 20, Universidad de los Andes, Venezuela, 1978.
24. H. W. MILNES, Convexity of Orlicz spaces, *Pacific J. Math.* **7** (1957), 1451–1483.
25. E. M. SEMENOV, Imbedding theorems for Banach function spaces of measurable functions, *Dokl. Akad. Nauk SSSR* **156** (1964), 1292–1295 [Russian] = *Soviet Math. Dokl.* **5** (1964), 831–834.
26. E. M. SEMENOV, A new interpolation theorem, *Funct. Anal. Appl.* **2** (1968), 158–168.
27. R. SHARPLEY, Spaces $\Lambda_{\alpha}(X)$ and interpolation, *J. Funct. Anal.* **11** (1972), 479–513.
28. R. SHARPLEY, Interpolation problems for compact operators, *Indiana Univ. Math. J.* **22** (1973), 965–984.
29. T. SHIMOGAKI, A note on norms of compression operators on function spaces, *Proc. Japan Acad.* **46** (1970), 239–242.
30. M. ZIPPIN, Interpolation of operators of weak type between rearrangement invariant function spaces. *J. Funct. Anal.* **7** (1971), 267–284.